DECOMPOSITION OF A LAYER OF VARIABLE MASS WITH PROGRESSIVE ENERGY RELEASE

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A self-similar solution with shock wave is found for the Lagrange equations of motion of a perfect gas in which energy is released. This solution is continued by means of a difference scheme.

This paper is concerned with a gasdynamic model of the process of sublimation of a wall located in a vacuum in the presence of energy release in the wall material. The simulation of the process is divided into two parts: 1) calculation of the dispersion of the gaseous products with simultaneous energy release, and 2) calculation of the gas flow after the energy sources have ceased to act. The function Q which describes the power of the sources is so selected that the decomposition of the wall proceeds at a constant rate. In this case, the first of the two problems mentioned above is self-similar, and its solution reduces to ordinary differential equations; the second problem employs the results of the solution of the first as initial data.

1. Formulation of the problem. We have to solve the system of equations describing the motion of a perfect gas in which energy is released:

$$\frac{\partial u}{\partial t}+\frac{\partial p}{\partial m}=0, \quad \frac{\partial V}{\partial t}-\frac{\partial u}{\partial m}=0,$$

$$V\frac{\partial p}{\partial t}+\gamma p\frac{\partial V}{\partial t}=(\gamma-1)Q. \tag{1}$$

On one side the gas is bounded by the sublimation surface; on the other its boundary is free (Fig. 1). At the free boundary

$$p=0, \quad \rho=0 \quad \text{at} \quad m=0. \tag{2}$$

The conditions at the sublimation surface include the mass balance equation and the temperature condition

$$\rho(w-u)=\rho_1 w, \quad p=R\rho T_1 \quad \text{at} \quad m=f(t). \tag{3}$$

The position of the phase interface is determined from the equation

$$\int_0^t Q(f(t),\ t)\,dt=\lambda+c_1(T_1-T_0)\equiv e_s. \tag{4}$$

The sublimation temperature T_1 is assumed known and independent of pressure. For many actual materials, this dependence is so weak that it can be neglected.

2. Self-similar regime. The initial instant of gas phase formation is denoted by t = 0. The point m = = 0, t = 0 is a singular point of the solution of problem (1)–(3); at this point, the parameters of the gas are not uniquely determined, since both boundaries which figure in the conditions of the problem pass through the point. This prompts us to seek a self-similar solution.

The dependence of Q on m and t is assumed to take the form

$$Q=Cm^{-\alpha}t^{\alpha-1}, \tag{5}$$

which for $0<\alpha<1$ expresses the decrease in the energy released with respect to mass and time. Then the determining parameters of the problem are

$$m, \quad t, \quad \rho_1, \quad RT_1, \quad e_s, \quad C, \quad \alpha, \quad \gamma.$$

Apart from m and t, there are only two constants with independent dimensions; consequently, for such energy release the problem has a self-similar solution.

From Eq. (4) with $\alpha>0$ it follows that the phase interface moves uniformly into the subliming material:

$$f(t)=Dt,$$

where

$$D=(\alpha e_s)^{-1/\alpha}C^{1/\alpha}$$

is the mass velocity of the interface. From the cor-

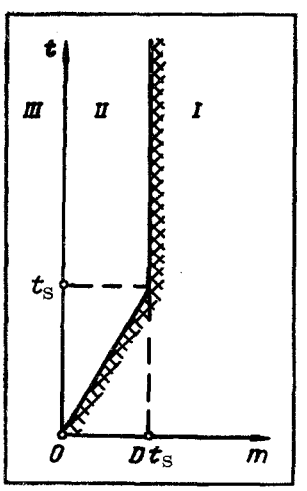

Fig. 1. Variation of the position of the gas-solid boundaries: I) solid; II) gas; III) vacuum.

responding expression for C we can obtain an expression for the function Q corresponding to motion of the interface at a given velocity D:

$$Q = \alpha e_s D^{\alpha} m^{-\alpha} t^{\alpha-1} \quad (\alpha > 0).$$

The choice of the independent dimensionless parameter ξ and the dependent parameters U, v, and P is determined by the relations

$$\xi = D^{-1} m t^{-1}, \quad u = u_0 U, \quad V = V_0 v, \quad p = p_0 P, \qquad (6)$$

where

$$u_0 = V_1 D, \quad p_0 = V_1 D^2, \quad V_0 = V_1.$$

Since $\partial/\partial t = -(\xi/t)(d/d\xi)$, $\partial/\partial m = (\xi/m)(d/d\xi)$, system (1) can be replaced by a system of three ordinary differential equations:

$$-\xi U' + P' = 0, \quad \xi v' + U' = 0,$$
$$vP' + \gamma P v' = -q\xi^{-\alpha-1}, \qquad (7)$$

where

$$q = \alpha(\gamma - 1) e_s V_1^{-2} D^{-2}.$$

In dimensionless form, the boundary conditions (2) and (3) become

$$P = 0, \quad v^{-1} = 0 \quad \text{at} \quad \xi = 0,$$
$$U = 1 - v, \quad Pv = \tau \quad \text{at} \quad \xi = 1. \qquad (8)$$

Here, $\tau = RT_1 p_0^{-1} V_0^{-1}$ is the dimensionless sublimation temperature.

3. Solution in the neighborhood of $\xi = 0$. Writing system (7) in the normal form,

$$v' = q\xi^{-\alpha-1}\Delta^{-1}, \quad P' = -q\xi^{1-\alpha}\Delta^{-1},$$
$$U' = -q\xi^{-\alpha}\Delta^{-1}, \quad \Delta = \xi^2 v - \gamma P, \qquad (9)$$

we find that the first two equations can be solved independently of the third. The point $\xi = 0$, $P = 0$, $v^{-1} = 0$ is a singular point of system (9). To construct the solution in the neighborhood of this point, we make the change of variables

$$s = \xi^2, \quad A = \xi^{1-\alpha/2} v^{-1}, \quad B = \gamma P v^{-1},$$

after which the first two equations take the form

$$\frac{dA}{ds} = \frac{qA^3 + (1 - \alpha/2) As (B - s)}{2s^2 (B - s)},$$
$$\frac{dB}{ds} = \frac{qA^2 (B + \gamma s)}{2s^2 (B - s)}. \qquad (10)$$

It is required to find a solution satisfying the conditions

$$A(0) = 0, \quad B(0) = 0.$$

Only trajectories entering the singular point at a certain tangent have physical meaning. With $A = ks$, $B = ls$ from (10) we obtain the following pairs of possible values for the direction numbers of the tangents k, l:

$$1) \quad k_1 = \left\{\frac{(1 + \alpha/2)[(1 + \alpha/2)\gamma - (1 - \alpha/2)]}{(1 - \alpha/2) q}\right\}^{1/2},$$
$$l_1 = \gamma \frac{1 + \alpha/2}{1 - \alpha/2};$$
$$2) \quad k_2 = -k_1, \quad l_2 = l_1;$$
$$3) \quad \text{any} \quad l_3 k_3^{-1}, \quad k_3^{-1} = 0;$$
$$4) \quad k_4 = 0, \quad l_4 \neq 0. \qquad (11)$$

Only the first pair gives a trajectory that can be interpreted as a solution of the problem in the neighborhood of the gas-vacuum interface. This solution has the form

$$v = \frac{\xi^{-1-\alpha/2}}{k_1}, \quad P = \frac{(1 + \alpha/2)\xi^{1-\alpha/2}}{(1 - \alpha/2) k_1},$$
$$U = -\frac{(1 + \alpha/2)\xi^{-\alpha}}{(\alpha/2) k_1}. \qquad (12)$$

4. Introduction of shock wave. Trajectory (12) does not satisfy the conditions on the boundary at $\xi = 1$. Accordingly we seek a discontinuous solution of the problem with a shock wave at $\xi = \xi_*$. The states of the gas to the left and right of the wave front are linked by the Hugoniot relations. Solved for the parameters to the right of the front, these relations take the form

$$P_+ = \frac{2}{\gamma + 1}\xi_*^2 v_- - \frac{\gamma - 1}{\gamma + 1} P_-,$$
$$U_+ = U_- + \frac{1}{\xi_*}(P_+ - P_-),$$
$$v_+ = v_- - \frac{1}{\xi_*}(U_+ - U_-). \qquad (13)$$

Here, the plus sign denotes values of the parameters to the right of the front, and the minus sign, values to the left.

To satisfy the two conditions at $\xi = 1$, it is necessary to have two parameters. As these parameters we can take ξ_* and α.

We introduce two auxiliary functions characterizing the degree of nonsatisfaction of the boundary conditions:

$$z_1 = 1 - Pv\tau^{-1}, \quad z_2 = 1 - v - U \quad (\text{at} \quad \xi = 1). \qquad (14)$$

The solution algorithm consists in finding the ξ_* and α that reduce z_1 and z_2 to zero. For this purpose we use the method of descent in the plane (ξ_*, α) with respect to the gradient of the function $\varphi(\xi_*, \alpha) = z_1^2 + z_2^2$.

5. Calculation of flow after discontinuance of the energy release. Calculation of the flow corresponding to cessation of energy release is of interest in connection with the study of the behavior of the shock wave. This calculation consists in the numerical solution of the boundary value problem for system (1) with $Q = 0$ in the half-strip $\{0 \leq m \leq Dt_s, \ t \geq t_s\}$, where t_s is the moment at which energy release is discontinued. The solution is obtained by means of a finite-

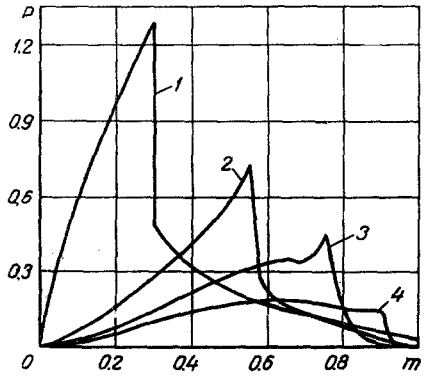

Fig. 2. Pressure profiles in the gas: 1) at $t/t_s = 1$; 2) 2.0311; 3) 3.1616; 4) 4.115.

difference scheme, which makes it possible to disregard discontinuities.

A check indicates displacement of the shock wave toward the wall. However, as a result of the intense motion of the entire mass of gas in the direction of the vacuum, a high degree of rarefaction develops in the region adjacent to the wall. This results in the pressure at the shock front falling to negligibly small values.

6. Example. Figure 2 presents the dimensionless pressure profiles obtained as a result of the numerical solution of a variant with the following starting data:

Calculation of the self-similar regime (in the first stage of the solution) gave values of ξ_* and α equal to 0.3034 and 0.6180, respectively. The corresponding pressure profile is represented by curve 1. Curves 2, 3, and 4, obtained in the second stage of the solution, illustrate the development of the shock wave. The time dependence of the characteristic pressures is given in the table.

Pressure at Shock Front and Pressure on Wall as Functions of Time

Curve	Dimensionless time t/t_s-1	Pressure at shock front	Pressure on wall
1	0	1.2776	$0.4073 \cdot 10^{-1}$
—	0.5002	0.8777	$0.2064 \cdot 10^{-1}$
2	1.0311	0.7264	$0.9556 \cdot 10^{-2}$
—	1.5776	0.5963	$0.5932 \cdot 10^{-2}$
3	2.1616	0.3584	$0.4136 \cdot 10^{-2}$
4	3.1150	0.1558	$0.2706 \cdot 10^{-2}$
—	5.8140	0.0302	$0.1298 \cdot 10^{-2}$
—	15.616	0.0053	$0.4004 \cdot 10^{-3}$

The example indicates that the action of the flow on the wall is mainly determined by the pressure developed when the source is functioning and that the aftereffect of the source is only slight.

NOTATION

m is the mass Lagrangian coordinate; t is the time; u is the gas velocity; p is the pressure; V is the specific volume of the gas; ρ is the gas density; T_1 is the sublimation temperature of the wall material; ρ_1 is the density of the wall material; c_1 is the specific heat of the wall material; R is the gas constant; w is the rate of decomposition of the wall; T_0 is the initial temperature of the wall; λ is the latent heat of sublimation; γ is the ratio of specific heats of the gas; $Q = Q(m, t)$ is the source function, expressing the amount of energy released per unit mass per unit time; $m = f(t)$ is the equation of the gas-solid interface.

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